# Deformed squeezed states in noncommutative phase space *

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**Abstract**

A deformed boson algebra is naturally introduced from studying quantum mechanics on noncommutative phase space in which both positions and momenta are noncommuting each other. Based on this algebra, corresponding intrinsic noncommutative coherent and squeezed state representations are constructed, and variances of single- and two-mode quadrature operators on these states are evaluated. The result indicates that in order to maintain Heisenberg's uncertainty relations, a restriction between the noncommutative parameters is required.

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## 1 Introduction

Recently there has been much interest in the study of physics in noncommutative space, mainly due to the noncommutative space is necessary for describing low-energy effective theory of a D-brane with a B-field background, but also the noncommutativity of space-time could play an important role in quantum gravity. Besides a lot of papers contributed to the study of the noncommutative field theory [1, 2], there are many works devoted to the study of various aspects of quantum mechanics on the noncommutative space with usual (commutative) time coordinate [3]-[5]. Although in string theory only the coordinate space exhibits a noncommutative structure, some authors have studied models in which a noncommutative geometry [6] defines the whole phase space [7, 8]. Noncommutativity between momenta arises naturally as a consequence of noncommutativity between coordinates, as momenta are defined to be the partial derivatives of the action with respect to the noncommutative coordinates [9].

The usual method to study the noncommutative quantum mechanics (NCQM) is using the Seiberg-Witten map to change a problem of NCQM into a corresponding problem of quantum mechanics on the commutative space. In the case of only the coordinate space is noncommutative, this method is consistent with the Weyl-Moyal correspondence which

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amounts to replacing the usual product by a star product in noncommutative space. This method, however, does not always work, for example, when both coordinates and momenta are noncommutative, i.e., on a noncommutative phase space, although using the Seiberg-Witten map one can write a Hamiltonian of NCQM in terms of the ordinary commutative coordinates and momenta variables, one has no way to get a well-defined Schrödinger equation which is consistent with the corresponding star product. Therefore, it is necessary to develop other new method to solve the quantum mechanical problems directly in the noncommutative phase space.

Our another motivation is to build some effective representations in NCQM. Since the noncommutativity between different coordinate (or momentum) component operators, there are no simultaneous eigenstates of these coordinate (or momentum) operators, therefore no exist ordinary coordinate (or momentum) representations in NCQM. However, in order to formulate quantum mechanics on the noncommutative space so that some dynamical problems can be solved, we do need some appropriate representations.

In this Letter we study how to construct noncommutative coherent and squeezed state representations for NCQM. We show that the coherent and squeezed state representations can be constructed equally well not only from the ordinary boson algebraic relations but also from a kind of deformed boson commutation relations which can be derived from a four-dimensional noncommutative phase space defined by the noncommutative parameters $\mu$ and $\nu$ in the following way

$$\begin{aligned} &[\hat{x},\,\hat{y}] = i\,\mu\,, && [\hat{p}_x,\,\hat{p}_y] = i\,\nu\,, \\ &[\hat{x},\,\hat{p}_x] = [\hat{y},\,\hat{p}_y] = i\,\hbar\,, && [\hat{x},\,\hat{p}_y] = [\hat{y},\,\hat{p}_x] = 0\,, \end{aligned} \tag{1.1}$$

where without loss of generality we have taken $\mu$, $\nu$ as positive real parameters. A deformed boson algebra was derived from such kind of deformed Heisenberg-Weyl algebra (1.1) under an assumption of maintaining Bose-Einstein statistics [10], (this assumption is equivalent to propose a direct proportionality between the noncommutative parameters $\mu$ and $\nu$), and some authors further investigated variances of the deformed single- and two-mode quadrature operators in the noncommutative phase space [11]-[13]. However, recently Bertolami and Rosa argued that there is no strong argument supporting such a direct proportionality relation [14]. In this Letter we derive the same deformed boson algebra (Eq.(2.2), see below) without using any additional assumption. All our analysis and calculation are performed directly in the noncommutative phase space and not by virtue of corresponding variables in the commutative space which appear via some kind of Seiberg-Witten map.

The work is organized as follows. In next section we use a simple method to derive the deformed boson algebra from the commutation relations in Eq.(1.1). Based on this deformed boson algebra we introduce two-mode displacement and squeeze operators in the noncommutative phase space and show that the two-mode squeeze operator induces a generalized Bogoliubov's transformation. In section 3 two-mode coherent and squeezed states are constructed on the noncommutative phase space and some basic properties of these states (such as inner product and overcompleteness) are given, which enable them to be effective representations. In section 4, variances of the deformed single- and two-mode quadrature operators in the noncommutative phase space are evaluated, and a constrain relation between the noncommutative parameters $\mu$ and $\nu$ is derived from the Heisenberg uncertainty relations. The last section devotes to some discussion and comment.

## 2 Deformed bosonic realization of noncommutative phase space

We start from Eq.(1.1) on the noncommutative phase space and introduce following deformed boson operators

$$\hat{a} = \frac{1}{\sqrt{2\hbar}} \left( \sqrt[4]{\frac{\nu}{\mu}}\, \hat{x} + i \sqrt[4]{\frac{\mu}{\nu}}\, \hat{p}_x \right), \qquad \hat{b} = \frac{1}{\sqrt{2\hbar}} \left( \sqrt[4]{\frac{\nu}{\mu}}\, \hat{y} + i \sqrt[4]{\frac{\mu}{\nu}}\, \hat{p}_y \right) \tag{2.1}$$

and their Hermitian conjugate operators $\hat{a}^\dagger$, $\hat{b}^\dagger$, which satisfy commutation relations

$$[\hat{a},\, \hat{a}^\dagger] = [\hat{b},\, \hat{b}^\dagger] = 1, \qquad [\hat{a},\, \hat{b}] = [\hat{a}^\dagger,\, \hat{b}^\dagger] = 0, \qquad [\hat{a},\, \hat{b}^\dagger] = -[\hat{b},\, \hat{a}^\dagger] = i\theta, \tag{2.2}$$

where $\theta = \sqrt{\mu\nu}/\hbar$. Here we would like to emphasize that the expressions in (2.1) only work for the case of $\mu$ and $\nu$ both nonzero, and this is just the situation we are considering. The algebraic relations in Eq.(2.2) are exactly the same in form as the deformed boson algebra in [10]. To our knowledge, this type of deformed boson commutation relations also appeared in early work by Caves and Schumaker [15]. Their quadrature-phase amplitudes satisfy the similar commutation relations as Eq.(2.2). When $\theta = 0$, Eq.(2.2) reduces to ordinary boson algebra.

An unitary two-mode displacement operator in the noncommutative phase space may be defined as

$$D(\alpha, \beta) = \exp(\alpha \hat{a}^\dagger + \beta \hat{b}^\dagger - \alpha^* \hat{a} - \beta^* \hat{b}), \tag{2.3}$$

with the important property

$$D(\alpha, \beta)^\dagger\, \hat{a}\, D(\alpha, \beta) = \hat{a} + \alpha + i\theta\beta, \qquad D(\alpha, \beta)^\dagger\, \hat{b}\, D(\alpha, \beta) = \hat{b} + \beta - i\theta\alpha. \tag{2.4}$$

Now we introduce a two-mode squeeze operator in the noncommutative phase space

$$\hat{S}(z) = \exp(z^* \hat{a}\hat{b} - z\hat{a}^\dagger \hat{b}^\dagger). \tag{2.5}$$

Inspection of the above operator shows that $\hat{S}(z)^\dagger = \hat{S}(z)^{-1} = \hat{S}(-z)$. Using Eq.(2.2) after straightforward calculation one can derive the following transformation ($z = r\, e^{i\varphi}$)

$$\begin{aligned} \hat{S}(z)\hat{a}\hat{S}(z)^\dagger &= (\cosh r\, \cosh r\theta)\hat{a} + i(\sinh r\, \sinh r\theta)\hat{b} \\ &\quad + e^{i\varphi}(\sinh r\, \cosh r\theta)\hat{b}^\dagger + ie^{i\varphi}(\cosh r\, \sinh r\theta)\hat{a}^\dagger, \\ \hat{S}(z)\hat{b}\hat{S}(z)^\dagger &= (\cosh r\, \cosh r\theta)\hat{b} - i(\sinh r\, \sinh r\theta)\hat{a} \\ &\quad + e^{i\varphi}(\sinh r\, \cosh r\theta)\hat{a}^\dagger - ie^{i\varphi}(\cosh r\, \sinh r\theta)\hat{b}^\dagger, \end{aligned} \tag{2.6}$$

which clearly generalizes well-known Bogoliubov's transformation [16] in commutative space to the noncommutative phase space, and of course, when $\theta = 0$, it reduces to the ordinary case. So we will call it generalized or deformed Bogoliubov's transformation. In fact, a $q$-deformed Bogoliubov's transformation for a pair of $q$-oscillators has been considered by A. Zhedanov [17], who has also shown that the $q$-deformed Bogoliubov's transformation is nonlinear and is related to Kravchuk's $q$-polynomials. For the later convenience, we denote the squeezed deformed boson operators $\hat{a}$ and $\hat{b}$ as

$$\hat{A} = \hat{S}(z)\, \hat{a}\, \hat{S}(z)^\dagger, \qquad \hat{B} = \hat{S}(z)\, \hat{b}\, \hat{S}(z)^\dagger. \tag{2.7}$$

Of course, the operators $\hat{A}$, $\hat{B}$ satisfy the same commutation relations as $\hat{a}$ and $\hat{b}$ do (Eq.(2.2)) because they are related each other by an unitary squeeze transformation.

## 3 Two-mode coherent and squeezed states in noncommutative phase space

Eq.(2.2) shows that the operators $\hat{a}$ and $\hat{b}$ are commuting each other and they have simultaneous eigenstates, i.e., coherent states. Starting from a vacuum state $|0\rangle$, which is normalized and destroyed by the deformed boson operators $\hat{a}$ and $\hat{b}$, and using the two-mode displacement operator (2.3), one can construct a two-mode coherent state in the noncommutative phase space

$$|\alpha,\beta\rangle = D(\alpha,\beta)|0\rangle, \tag{3.1}$$

which satisfies

$$\hat{a}|\alpha,\beta\rangle = (\alpha + i\theta\beta)|\alpha,\beta\rangle, \qquad \hat{b}|\alpha,\beta\rangle = (\beta - i\theta\alpha)|\alpha,\beta\rangle, \tag{3.2}$$

respectively. The inner product of two such coherent states is easily to get

$$\begin{aligned}\langle\alpha',\beta'|\alpha,\beta\rangle \;=\; \exp\Big\{ &-\frac{1}{2}\big(|\alpha|^2+|\beta|^2+|\alpha'|^2+|\beta'|^2\big)+\alpha'^*\alpha+\beta'^*\beta \\ &+\frac{i\theta}{2}(\beta^*\alpha-\alpha^*\beta+\beta'^*\alpha'-\alpha'^*\beta')+i\theta(\alpha'^*\beta-\beta'^*\alpha)\Big\},\end{aligned} \tag{3.3}$$

which means that the two-mode coherent states are normalized but not orthogonal to each other, and besides, they are over-complete. The corresponding resolution of the identity is

$$(1-\theta^2)\int \frac{d^2\alpha\, d^2\beta}{\pi^2}|\alpha,\beta\rangle\,\langle\alpha,\beta| = 1, \tag{3.4}$$

where the integral is over two complex planes, and $d^2\alpha = d\mathrm{Re}(\alpha)\, d\mathrm{Im}(\alpha)$, $d^2\beta = d\mathrm{Re}(\beta)\, d\mathrm{Im}(\beta)$.

In the noncommutative phase space, a two-mode squeezed state $|\alpha,\beta;z\rangle$ can be defined as

$$|\alpha,\beta;z\rangle = \hat{S}(z)|\alpha,\beta\rangle, \tag{3.5}$$

which is simultaneous eigenstate of the squeezed boson operators $\hat{A}$ and $\hat{B}$,

$$\hat{A}|\alpha,\beta;z\rangle = (\alpha + i\theta\beta)|\alpha,\beta;z\rangle, \qquad \hat{B}|\alpha,\beta;z\rangle = (\beta - i\theta\alpha)|\alpha,\beta;z\rangle. \tag{3.6}$$

Multiplying Eq.(3.4) by $\hat{S}(z)$ from the left-side and by $\hat{S}^\dagger(z)$ from the right-side, one has

$$(1-\theta^2)\int \frac{d^2\alpha\, d^2\beta}{\pi^2}|\alpha,\beta;z\rangle\,\langle\alpha,\beta;z| = 1, \tag{3.7}$$

which means that the two-mode squeezed states also satisfy the resolution of identity. Thus we have coherent representation and squeezed representation in the noncommutative phase space, and any state in this space can be expanded in terms of $|\alpha,\beta\rangle$ or $|\alpha,\beta;z\rangle$.

Using the algebra (2.2) one can straightforwardly evaluate the wavefunction of the two-

mode squeezed state $|\alpha,\beta;z\rangle$ in the two-mode coherent state representation, i.e.,

$$
\begin{aligned}
&\langle\alpha',\beta'|\alpha,\beta;z\rangle = \Big(\cosh(r(1+\theta))\cosh(r(1-\theta))\Big)^{-1/2}\\
&\times\exp\Big(-\frac{1}{2}\big(|\alpha|^2+|\beta|^2+|\alpha'|^2+|\beta'|^2+i\theta(\alpha^*\beta-\beta^*\alpha+\alpha'^*\beta'-\beta'^*\alpha')\big)\Big)\\
&\times\exp\Big\{\frac{(1+\theta)(\alpha'^*\alpha+\beta'^*\beta+i\alpha'^*\beta-i\beta'^*\alpha)}{2\cosh(r(1+\theta))}+\frac{(1-\theta)(\alpha'^*\alpha+\beta'^*\beta-i\alpha'^*\beta+i\beta'^*\alpha)}{2\cosh(r(1-\theta))}\Big\}\\
&\times\exp\Big\{-ie^{-i\varphi}\Big(\frac{1+\theta}{4}\tanh(r(1+\theta))(\alpha+i\beta)^2-\frac{1-\theta}{4}\tanh(r(1-\theta))(\alpha-i\beta)^2\Big)\\
&-ie^{i\varphi}\Big(\frac{1+\theta}{4}\tanh(r(1+\theta))(\alpha'^*-i\beta'^*)^2-\frac{1-\theta}{4}\tanh(r(1-\theta))(\alpha'^*+i\beta'^*)^2\Big)\Big\}.
\end{aligned}
\tag{3.8}
$$

Obviously when $z=0$, the above expression will reduce to Eq.(3.3). Having (3.8) and using (3.4), one can evaluate another important inner product of two arbitrary such squeezed states $\langle\alpha',\beta';z'|\alpha,\beta;z\rangle$.

## 4 Heisenberg uncertainty relations

Obviously, the commutation relations (1.1) lead to the following uncertainty relations

$$
\triangle\hat{x}\triangle\hat{y}\geqslant\frac{\mu}{2},\qquad \triangle\hat{p}_x\triangle\hat{p}_y\geqslant\frac{\nu}{2},\qquad \triangle\hat{x}\triangle\hat{p}_x\geqslant\frac{\hbar}{2},\qquad \triangle\hat{y}\triangle\hat{p}_y\geqslant\frac{\hbar}{2}. \tag{4.1}
$$

The first two uncertainty relations show that measurements of positions and momenta in both directions $x$ and $y$ are not independent. Taking into account the fact that $\mu$ and $\nu$ have dimensions of $(length)^2$ and $(momentum)^2$ respectively, then $\sqrt{\mu}$ and $\sqrt{\nu}$ define fundamental scales of length and momentum which characterize the minimum uncertainties possible to achieve in measuring these quantities. One expects these fundamental scales to be related to the scale of the underlying field theory (possible the string scale), and thus to appear as small corrections at the low-energy level or quantum mechanics.

Now one can evaluate the variances of the single-mode quadrature operators $\hat{x}$ and $\hat{p}_x$ on the state $|\alpha,\beta;z\rangle$ and get

$$
\begin{aligned}
(\triangle\hat{x})^2_z &= \langle\alpha,\beta;z|\hat{x}^2|\alpha,\beta;z\rangle-\langle\alpha,\beta;z|\hat{x}|\alpha,\beta;z\rangle^2\\
&= \frac{\hbar}{2}\sqrt{\frac{\mu}{\nu}}\Big(\cosh 2r(\cosh 2r\theta+\sin\varphi\sinh 2r\theta)\\
&\qquad +\ \theta\sinh 2r(\sinh 2r\theta+\sin\varphi\cosh 2r\theta)\Big),\\
(\triangle\hat{p}_x)^2_z &= \langle\alpha,\beta;z|\hat{p}_x^2|\alpha,\beta;z\rangle-\langle\alpha,\beta;z|\hat{p}_x|\alpha,\beta;z\rangle^2\\
&= \frac{\hbar}{2}\sqrt{\frac{\nu}{\mu}}\Big(\cosh 2r(\cosh 2r\theta-\sin\varphi\sinh 2r\theta)\\
&\qquad +\theta\sinh 2r(\sinh 2r\theta-\sin\varphi\cosh 2r\theta)\Big),
\end{aligned}
\tag{4.2}
$$

where the subscript $z$ on the left-hand side of Eq.(4.2) means the variance is for the state $|\alpha,\beta;z\rangle$. Furthermore, one has

$$
\sqrt{\frac{\nu}{\mu}}(\triangle\hat{x})^2_z=\sqrt{\frac{\mu}{\nu}}(\triangle\hat{p}_y)^2_z\,,\qquad \sqrt{\frac{\nu}{\mu}}(\triangle\hat{y})^2_z=\sqrt{\frac{\mu}{\nu}}(\triangle\hat{p}_x)^2_z\,. \tag{4.3}
$$

These results show that the variances of the single-mode quadrature operators are independent on the complex numbers $\alpha$, $\beta$ in the squeezed state $|\alpha,\beta;z\rangle$. They are dependent only on the squeezing parameter $z$ and the noncommutative parameter $\theta$.

Eq.(4.2) leads to

$$(\triangle\hat{x})_z^2(\triangle\hat{p}_x)_z^2=\frac{\hbar^2}{4}\Big[\cosh^2 2r\,\big(\cosh^2 2r\theta-\sin^2\varphi\,\sinh^2 2r\theta\big)+\frac{1}{2}\theta\cos^2\varphi\,\sinh 4r\,\sinh 4r\theta \\ +\,\theta^2\sinh^2 2r\,\big(\sinh^2 2r\theta-\sin^2\varphi\,\cosh^2 2r\theta\big)\Big]. \quad (4.4)$$

One will find that for $\varphi=|\pi|/2$, $(\triangle\hat{x})_z^2(\triangle\hat{p}_x)_z^2$ reaches its minimum,

$$\begin{aligned}\min\big((\triangle\hat{x})_z^2(\triangle\hat{p}_x)_z^2\big) &= \frac{\hbar^2}{4}\big(1+(1-\theta^2)\sinh^2 2r\big)\\ &= \frac{\hbar^2}{4}\big(1+(1-\frac{\mu\nu}{\hbar^2})\sinh^2 2r\big).\end{aligned} \quad (4.5)$$

From this expression we see that there exists a natural constraint for the noncommutative parameters $\mu$ and $\nu$, i.e., only when $\mu\nu\leqslant\hbar^2$ the Heisenberg uncertainty relation $\triangle\hat{x}\triangle\hat{p}_x\geqslant\hbar/2$ is satisfied, and if $\mu\nu=\hbar^2$, $\triangle\hat{x}\triangle\hat{p}_x=\hbar/2$. The same is for $\hat{y}$ and $\hat{p}_y$.

Also for $\varphi=|\pi|/2$ one has

$$\begin{aligned}\min\big((\triangle\hat{x})_z^2(\triangle\hat{y})_z^2\big) &= \frac{\hbar^2\mu}{4\nu}\big(1+(1-\frac{\mu\nu}{\hbar^2})\sinh^2 2r\big),\\ \min\big((\triangle\hat{p}_x)_z^2(\triangle\hat{p}_y)_z^2\big) &= \frac{\hbar^2\nu}{4\mu}\big(1+(1-\frac{\mu\nu}{\hbar^2})\sinh^2 2r\big).\end{aligned} \quad (4.6)$$

It is easy to find that in order to have the Heisenberg uncertainty relations $\triangle\hat{x}\triangle\hat{y}\geqslant\mu/2$ and $\triangle\hat{p}_x\triangle\hat{p}_y\geqslant\nu/2$, it also needs $\mu\nu\leqslant\hbar^2$, and for $\mu\nu=\hbar^2$, the Heisenberg uncertainty relations become equalities.

It is interesting to compare the variances of these single-mode quadrature operators on the state $|\alpha,\beta;z\rangle$ with ones on the coherent state $|\alpha,\beta\rangle$. One can simply find that the variances of $\hat{x}$ on the state $|\alpha,\beta\rangle$ is

$$(\triangle\hat{x})^2=\langle\alpha,\beta|\hat{x}^2|\alpha,\beta\rangle-\langle\alpha,\beta|\hat{x}|\alpha,\beta\rangle^2=\frac{\hbar}{2}\sqrt{\frac{\mu}{\nu}}\,, \quad (4.7)$$

and similarly

$$(\triangle\hat{p}_x)^2=\frac{\hbar}{2}\sqrt{\frac{\nu}{\mu}}\,,\qquad(\triangle\hat{y})^2=(\triangle\hat{x})^2\,,\qquad(\triangle\hat{p}_y)^2=(\triangle\hat{p}_x)^2\,. \quad (4.8)$$

These relations show that the last two Heisenberg's relations in (4.1) reach their minimums on the coherent states, however, the first two relations in (4.1) work also only for $\mu\nu\leqslant\hbar^2$. Furthermore, from (4.2) and (4.7), one has

$$(\triangle\hat{x})_z^2 = \Big(\cosh 2r(\cosh 2r\theta+\sin\varphi\sinh 2r\theta) \\ +\,\theta\sinh 2r(\sinh 2r\theta+\sin\varphi\cosh 2r\theta)\Big)(\triangle\hat{x})^2, \quad (4.9)$$

and also

$$(\triangle\hat{p}_x)_z^2 = \Big(\cosh 2r(\cosh 2r\theta-\sin\varphi\sinh 2r\theta) \\ +\,\theta\sinh 2r(\sinh 2r\theta-\sin\varphi\cosh 2r\theta)\Big)(\triangle\hat{p}_x)^2. \quad (4.10)$$

If $\theta = 0$, (4.9) and (4.10) give

$$(\triangle\hat{x})^2_z > (\triangle\hat{x})^2\,, \qquad (\triangle\hat{p}_x)^2_z > (\triangle\hat{p}_x)^2 \tag{4.11}$$

for nonzero squeezing parameter $z$, which mean that on the squeezed state $|\alpha, \beta; z\rangle$ the corresponding Heisenberg uncertainty relation cannot reach its minimum. Only due to the existence of nonzero $\theta$, the case

$$(\triangle\hat{x})^2_z \leqslant (\triangle\hat{x})^2, \qquad (\triangle\hat{p}_x)^2_z \leqslant (\triangle\hat{p}_x)^2 \tag{4.12}$$

may occur for some ranges of the parameter $z$ and the two-mode squeezed state $|\alpha, \beta; z\rangle$ may exhibit some squeezing effects for these cases.

It is worth while to point out that one can also choose, instead of (2.1), the following operators [1]

$$\begin{aligned} a &= (\sqrt{2\hbar})^{-1}\Big(\kappa - \frac{\mu\nu}{4\kappa\hbar^2}\Big)^{-1}\Big(\kappa\hat{x} + \frac{\mu}{2\hbar}\hat{p}_y + i\big(-\frac{\nu}{2\kappa\hbar}\hat{y} + \hat{p}_x\big)\Big)\,, \\ b &= (\sqrt{2\hbar})^{-1}\Big(\kappa - \frac{\mu\nu}{4\kappa\hbar^2}\Big)^{-1}\Big(\kappa\hat{y} - \frac{\mu}{2\hbar}\hat{p}_x + i\big(\frac{\nu}{2\kappa\hbar}\hat{x} + \hat{p}_y\big)\Big)\,, \end{aligned} \tag{4.13}$$

to construct a basis of coherent or squeezed states to be used for calculation of the expectation values. It is easy to show that, when $\kappa = \big(1 + \sqrt{1 - \frac{\mu\nu}{\hbar^2}}\,\big)/2$, the operators $a$, $b$ and their Hermitian conjugates $a^\dagger$, $b^\dagger$ satisfy ordinary boson commutation relations

$$[a,\, a^\dagger] = [b,\, b^\dagger] = 1, \qquad [a,\, b] = [a,\, b^\dagger] = 0. \tag{4.14}$$

Now, with these ordinary boson operators, one can equally well construct a two-mode overcomplete basis of coherent states as well as squeezed states. Of course, the basic expressions for these states are the standard ones. Starting from these ordinary coherent states or ordinary squeezed states, one can readily evaluate the variances for the coordinate and momentum operators $\hat{x}$, $\hat{y}$, $\hat{p}_x$, $\hat{p}_y$ and get the same restriction $\mu\nu \leqslant \hbar^2$ only from the condition that the parameter $\kappa$ has to be real in order to be able to introduce the above operators $a$ and $b$. Also the expressions in (4.14) work for any noncommutative parameters $\mu$ and $\nu$, no matter they are zero or nonzero. Needless to say, there are many possible choice of such kind of maps.

Now we turn to discuss the two-mode quadrature operators which are defined as

$$\begin{aligned} \hat{X} &= \frac{\hat{x} + \hat{y}}{2} = \sqrt{\frac{\hbar}{2}}\sqrt[4]{\frac{\mu}{\nu}}\,\frac{\hat{a} + \hat{b} + \hat{a}^\dagger + \hat{b}^\dagger}{2}\,, \\ \hat{P} &= \frac{\hat{p}_x + \hat{p}_y}{2} = \sqrt{\frac{\hbar}{2}}\sqrt[4]{\frac{\nu}{\mu}}\,\frac{\hat{a} + \hat{b} - \hat{a}^\dagger - \hat{b}^\dagger}{2i} \end{aligned} \tag{4.15}$$

and satisfy the following commutation relation $[\hat{X}, \hat{P}] = i\hbar/2$. Similarly to the case of the single-mode quadrature operators, one may derive the variances of $\hat{X}$ and $\hat{P}$ on the two-mode squeezed state $|\alpha, \beta; z\rangle$

$$\begin{aligned} (\triangle\hat{X})^2_z &= \frac{\hbar}{4}\sqrt{\frac{\mu}{\nu}}\big(\cosh 2r\theta(\cosh 2r - \cos\varphi \sinh 2r) \\ &\qquad + \theta \sinh 2r\theta(\sinh 2r - \cos\varphi \cosh 2r)\big), \\ (\triangle\hat{P})^2_z &= \frac{\hbar}{4}\sqrt{\frac{\nu}{\mu}}\big(\cosh 2r\theta(\cosh 2r + \cos\varphi \sinh 2r) \\ &\qquad + \theta \sinh 2r\theta(\sinh 2r + \cos\varphi \cosh 2r)\big), \end{aligned} \tag{4.16}$$

[1] Authors thank one of the referees to point out this.

which lead to

$$(\triangle\hat{X})_z^2(\triangle\hat{P})_z^2=\frac{\hbar^2}{16}\Big\{(\cosh^2 2r-\cos^2\varphi\,\sinh^2 2r)+\frac{\theta}{2}\sin^2\varphi\,\sinh 4r\,\sinh 4r\theta$$
$$+\big[(\cosh^2 2r+\theta^2\sinh^2 2r)-\cos^2\varphi\,(\theta^2\cosh^2 2r+\sinh^2 2r)\big]\sinh^2 2r\theta\Big\}. \quad (4.17)$$

When $\varphi=0$, (4.17) reaches its minimum

$$\min\big((\triangle\hat{X})_z^2(\triangle\hat{P})_z^2\big)=\frac{\hbar^2}{16}\big(1+(1-\theta^2)\sinh^2 2r\theta\big), \quad (4.18)$$

which also shows that the Heisenberg uncertainty relation $(\triangle\hat{X})_z(\triangle\hat{P})_z\geqslant\hbar/4$ requires $\mu\nu\leqslant\hbar^2$ and only for $\mu\nu=\hbar^2$ the uncertainty relation becomes an equality.

## 5 Discussion and conclusion

In this Letter we introduce the deformed boson operators $\hat{a}$, $\hat{b}$ in (2.1) and their Hermitian conjugate operators $\hat{a}^\dagger$, $\hat{b}^\dagger$, which satisfy the deformed boson algebra (2.2). Based on the deformed boson algebra, we construct deformed two-mode coherent states and squeezed states on the noncommutative phase space and show their inner products and overcompleteness properties, which enable these states form corresponding effective representations in the noncommutative phase space. Then we evaluate the variances of the single- and two-mode quadrature operators on the two-mode squeezed states $|\alpha,\beta;z\rangle$ and investigate corresponding Heisenberg uncertainty relations. From these variances, we find that there exists a constraint between the noncommutative parameters $\mu$ and $\nu$, and when $\mu\nu=\hbar^2$, all the Heisenberg inequalities become equalities, which is in agreement with [18]. Our analysis and calculation are performed on the noncommutative phase space straightforwardly, without depending on any variables on the commutative space.

It should be pointed out that when considering a concrete Hamiltonian, for example, the Hamiltonian of two-dimensional isotropic harmonic oscillator, $\hat{H}(\hat{x},\,\hat{p})=\frac{1}{2m}(\hat{p}_x^2+\hat{p}_y^2)+\frac{m\omega^2}{2}(\hat{x}^2+\hat{y}^2)$, in order to maintain the physical meaning of $\hat{a}$, $\hat{b}$, $\hat{a}^\dagger$ and $\hat{b}^\dagger$ , i.e., to keep the relations among $(\hat{a},\hat{b},\hat{a}^\dagger,\hat{b}^\dagger)$ and $(\hat{x},\hat{y},\hat{p}_x,\hat{p}_y)$ having the same formulations as the ones in commutative space, one must let the model parameters $m$ and $\omega$ satisfy a relation $m^2\omega^2=\mu/\nu$, which is equivalent to propose a direct proportionality between the noncommutative parameters $\mu$ and $\nu$ . In our opinion, the parameters $\mu$ and $\nu$ reflect the intrinsic noncommutativity between positions and momenta respectively, (like as the Planck constant encodes the noncommutativity of position and momentum,) which should be independent on the concrete physical model.

We would like also to emphasize that the coherent state representation and the squeezed state representation in the noncommutative phase space are constructed in this Letter. Besides this, we can also construct continuum entangled state representation for the NCQM. These constructed representations will certainly play important roles in studying physical problems on the noncommutative phase space. The related result will be reported in a following separated paper.

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